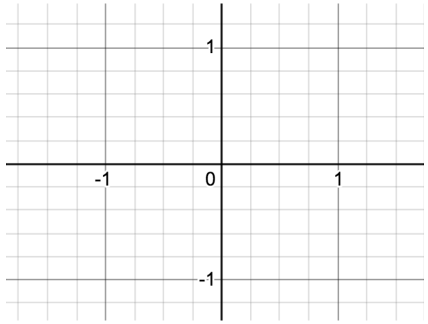
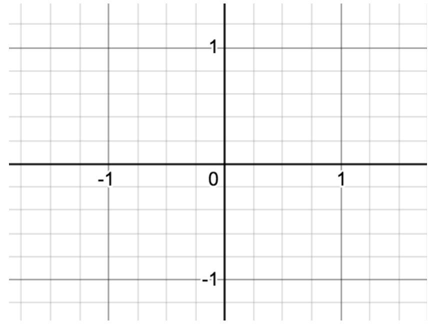
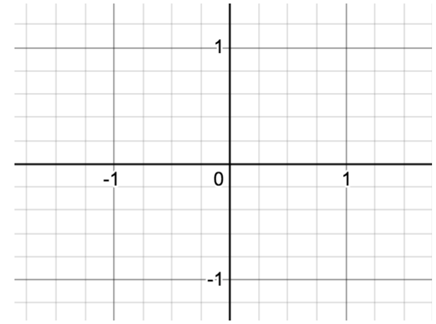
14. 1 Functions of Two or More Variables

Recall functions of ONE variable: $f: R \to R$







Functions of Two Variables: $f: R^2 \to R$

> **Definition** A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

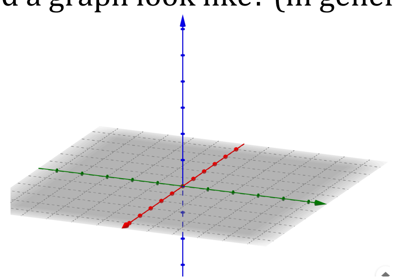
Example: $f(x,y) = \ln(x^2 - y)$

(x,y)	f(x,y)

Compute functional values

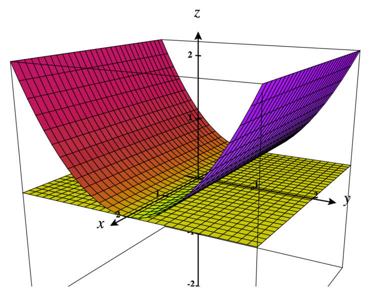
What is the domain?

What would a graph look like? (in general, we will look at more specific methods later)

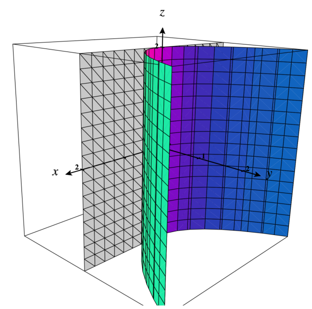


Other orientations for functions

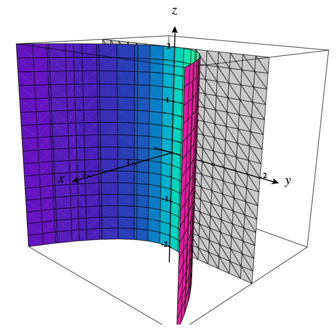
z=f(x,y)



y=f(x,z)

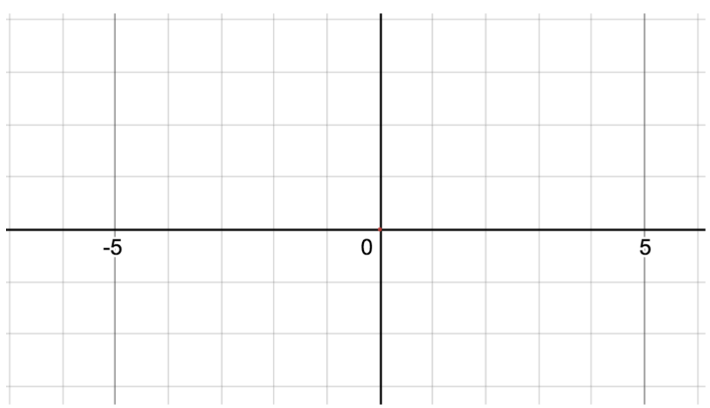


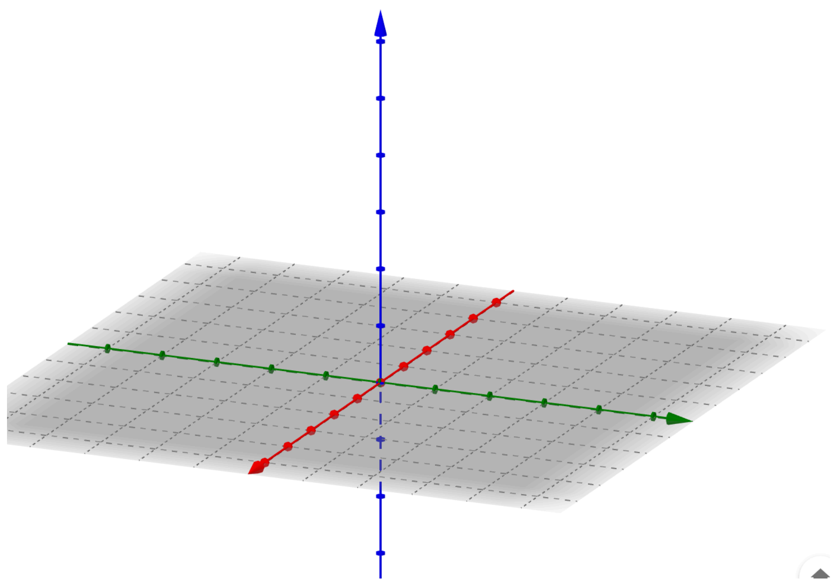
x=f(y,z)



Graphing functions of two variables:

Sketch the graph of $f(x,y)=\frac{1}{4}x^2+y^2$





Note: Level curves can provide very useful physical information about the function even if the goal is not a graph.

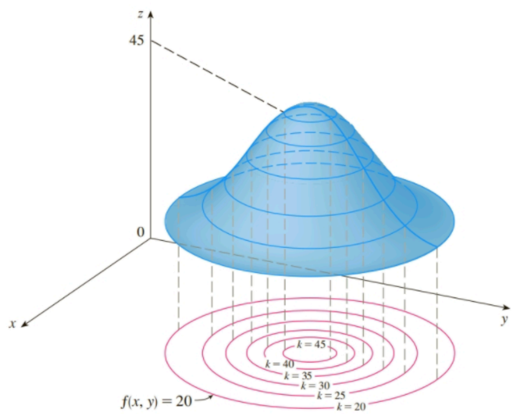


FIGURE 11

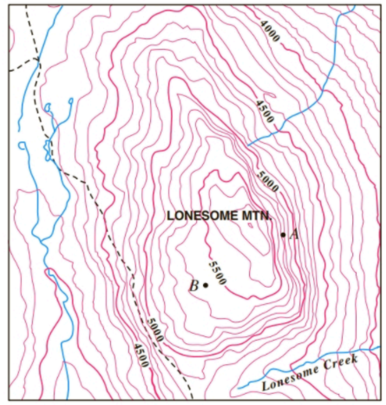
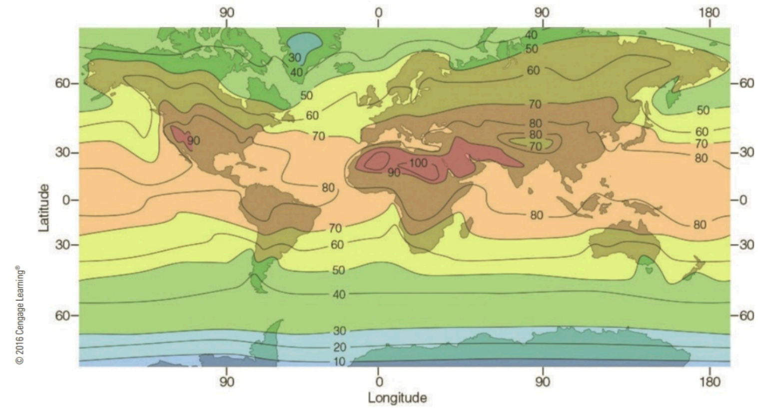


FIGURE 12



Functions of 3 variables

Definition similar, but domain is a set in ________________.

Example: $w = f(x,y,z) = \sqrt{100 - x^2 - y^2 - z^2}$

Find: $f(0,0,0)=$____________________ $f(1,2,0)=$____________________ $f(1,2,3)=$____________________

Domain:

Graph: How would be "graph" $f(0,0,0) = 1$

Level surfaces:

remains fixed.

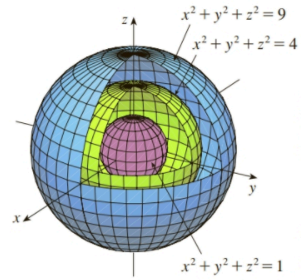


FIGURE 21

EXAMPLE 15 Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

SOLUTION The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geqslant 0$. These form a family of concentric spheres with radius $\sqrt{k}$. (See Figure 21.) Thus, as (x, y, z) varies over any sphere with center O, the value of $f(x, y, z)$ remains fixed. ■

Functions of any number of variables can be considered. A **function of *n* variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples. For example, if a company uses n different ingredients in making a food product, c_i is the cost per unit of the ith ingredient, and x_i units of the ith ingredient are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, \ldots, x_n$:

14.2 Limits – We cover this lightly

Recall limit for f(x) (1.7)

2 Precise Definition of a Limit Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

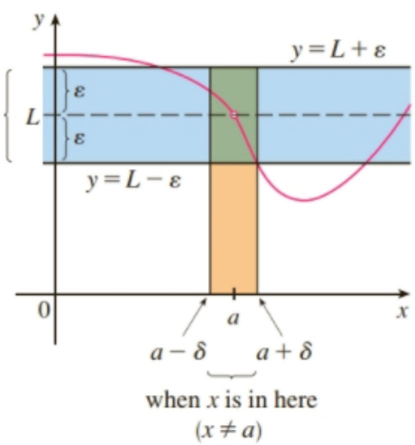
Extends in a logical way to f(x,y)

1 Definition Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$



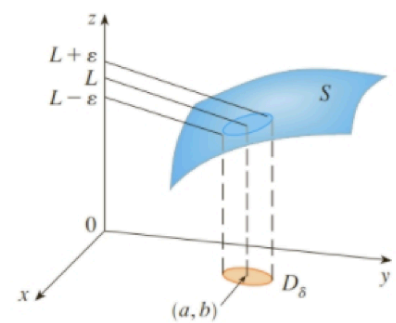


FIGURE 2

Chapter 14 Multivariable Functions

How do we compute limits? (see graphs Math 5C page "Limits" https://www.geogebra.org/classic/ntcdb2mt)
Most of the functions we deal with are continuous on their domain, so to evaluate a limit, we just evaluate the function

$$\lim_{(x,y)\to(1,1)}\left(\frac{x^2}{10}+y^2\right)=\_\_\_\_\_\_\_\_\_\_$$

4 Definition A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

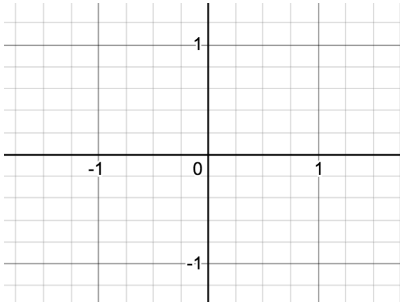
But if f is not continuous at (a,b) but is instead indeterminate, we do one of three things.

1) Algebraic manipulation

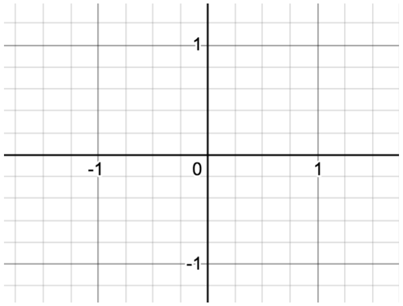
$$\lim_{(x,y)\to(0,0)}\frac{x^4-y^4}{x^2-y^2}=$$

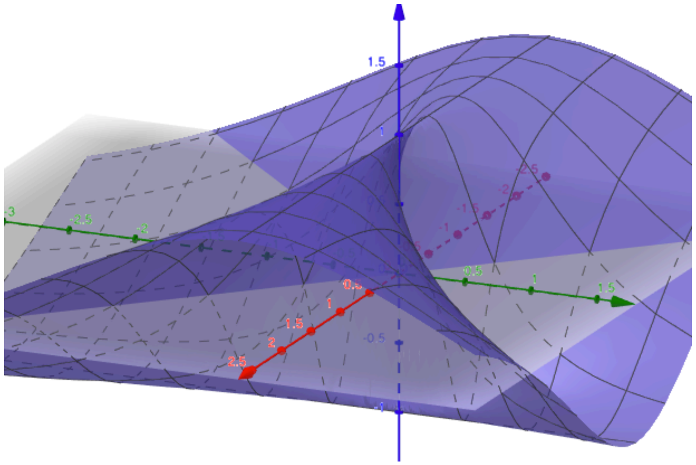
$$\lim_{(x,y)\to(0,0)}(x^2+y^2)\ln(x^2+y^2)$$

2) Prove the limit does not exist by considering different paths. (Recall $\lim_{x \to 0} \frac{|x|}{x}$)



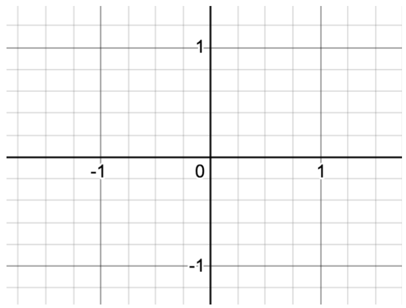
Example: $\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$



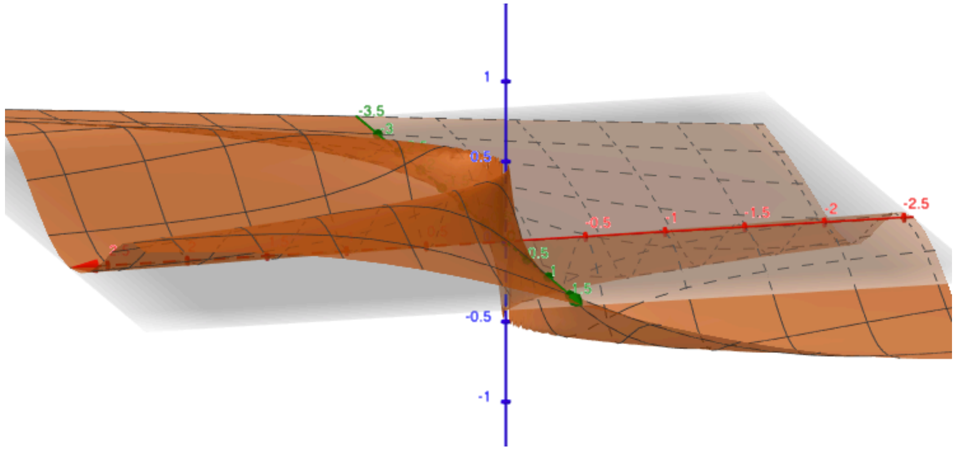


2) Consider different paths (cont'd)

Example: $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$



What is the moral of the story on this example? ____

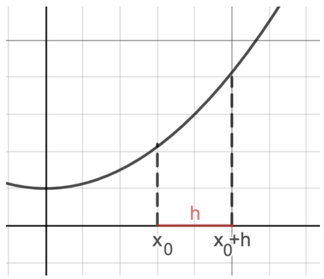


3) Prove using the delta epsilon definition of limit or squeeze theorem. (You won't be tested on these.)

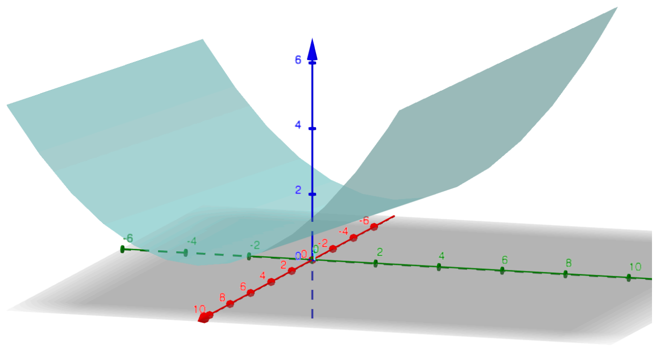
Example: $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2}$

Intro to 14.6 and 14.3: Development of the derivative of f(x,y)

Derivative of f(x)



Derivative of f(x,y)



Move a distance h from a given point x_0.

New point:

Move a distance h from a given point (x_0, y_0) in the direction of unit vector $\vec{u} = \langle a, b \rangle$

New point:

Average Rate of Change at x_0

$$\frac{\Delta output}{\Delta input} = \frac{rise}{run} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Average Rate of Change at (x_0, y_0) in the direction of unit vector $\vec{u} = \langle a, b \rangle$

$$\frac{\Delta output}{\Delta input} = \frac{rise}{run} = \frac{\Delta z}{} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Instantaneous Rate of Change at x_0

$f'(x_0) =$

Inst. Rate of Change at (x_0, y_0)in the direction of unit vector $\vec{u} = \langle a, b \rangle$

The general derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The general derivative:

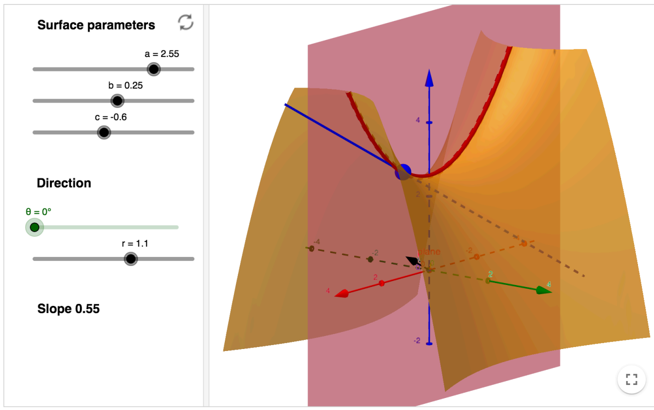
$$D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}$$

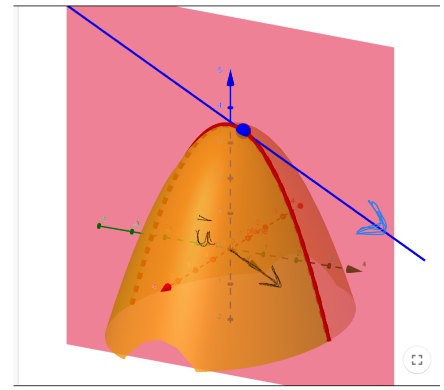
Understanding the derivative $D_{\vec{u}} f(x,y)$

Directional Derivatives

Author: Joseph Manthey

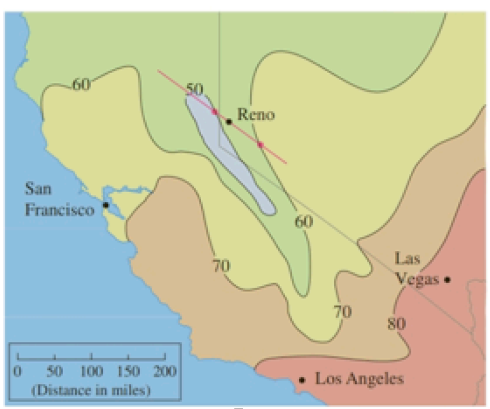
Topic: Derivative





https://www.geogebra.org/m/tZgrSxQ4#material/Trws2PBm

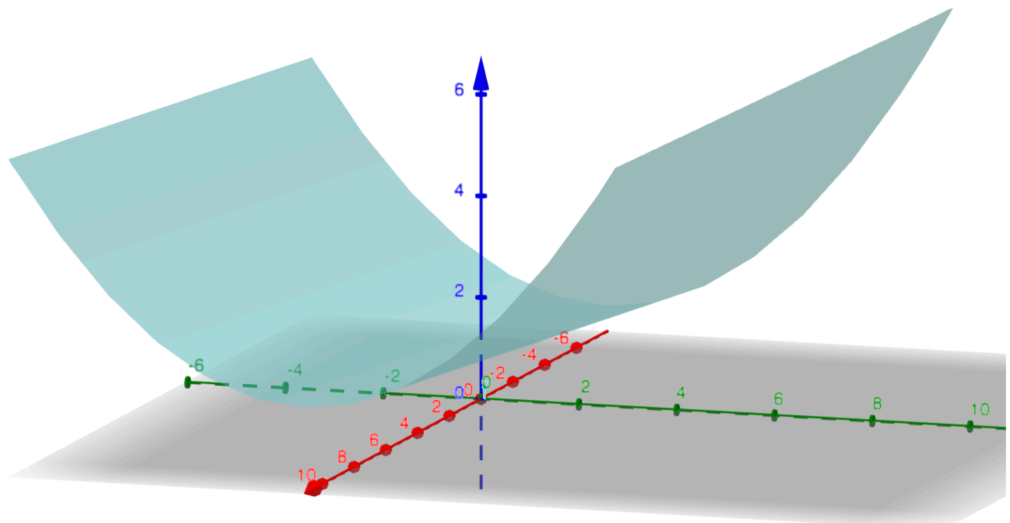
EXAMPLE 1 Use the weather map in Figure 1 to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.



14.3 Partial Derivatives

For now, though we should understand the meaning of the derivative $D_{\vec{u}} f(x,y)$ we are unable to compute it.

In this section, we consider to special cases of the derivative that we WILL be able to compute.



$\vec{u} = \vec{i} = <1,0>$

Movement in the ____________________ direction

y=__________________ (constant)

_________ $= D_{\vec{u}} f(x,y) =$

$\vec{u} = \vec{j} = <0,1>$

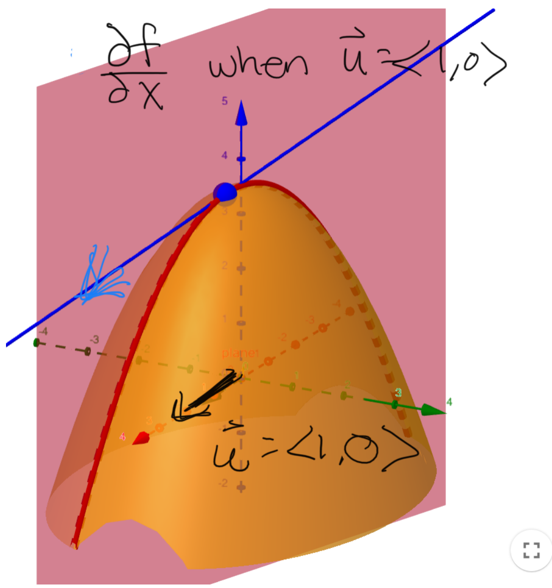
Movement in the ____________________ direction

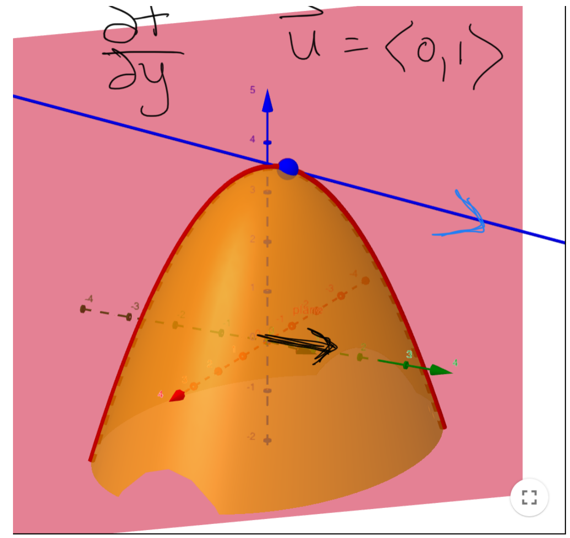
x=__________________ (constant)

_________ $= D_{\vec{u}} f(x,y) =$

Notation:

$$\frac{\partial f}{\partial x} = f_x \qquad\qquad \frac{\partial f}{\partial y} = f_y$$





Another illustration of partial derivatives on the 5C page: https://www.geogebra.org/m/RtISr7GW#material/gsypFXHC

Computing Partial Derivatives:

To compute $\frac{\partial f}{\partial x}$, we treat ________________ as a constant. To compute $\frac{\partial f}{\partial y}$, we treat ________________ as a constant

Example:

$f(x,y) = x^2 \ln y$ Find : $\frac{\partial f}{\partial x}$ f_y

$\left.\frac{\partial f}{\partial x}\right|_{(4,2)}$ $f_y(3,1)$

Differentiation extends to R[3], with the additional partial derivative corresponding to the positive z direction, $\vec{u} = <0,0,1>$

Example: Suppose $T(x,y,z) = \frac{100}{x^2+y^2+z^2}$ is the Temperature in °F at point (x,y,z). Find and interpret $T_z(1,2,3)$.

Approximating partial derivatives when discrete data or contour map given, no f(x,y)

Discrete Example: from pg 952 Given $f(T,H)$

Table 1 Heat index I as a function of temperature and humidity

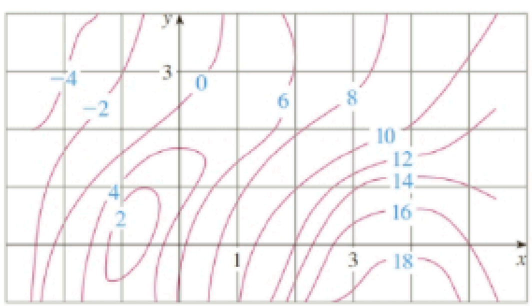
Relative humidity (%)

Actual temperature (°F)

T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Find: $f(70,96)$ $\qquad$ $f_H(60,96)$

Contour map example



Find: $f(2,0)$ $\qquad$ $f_x(2,0)$ $\qquad$ $\frac{\partial f}{\partial y}(2,0)$

m

Implicit Differentiation

For the surface $x^2+y^2+z^2=1$, find $\frac{\partial z}{\partial y}$ at the point $\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)$

Higher Order Partial Derivatives

$$f(x,y) = x^3y^4$$

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

14.5 – Chain Rules

Notation: If f is a function of ONE variable only, we use d. So if y=f(x), we say $\frac{dy}{dx}$

If f is a function of MORE than one variable, we use ∂ So if z=f(x,y), we say $\frac{\partial z}{\partial x}$

Two versions of Chain Rule

1) f is a function of more than one variable where each of those variables is a function of one variable only, so f is ultimately dependant on ONE variable.

Example: $z = x^2 y$, with $\begin{cases} x = t^2 \\ y = t^3 \end{cases}$

2 The Chain Rule (Case 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

2) f is a function of more than one variable where each of those variables is also a function of more than one variable , so f is ultimately dependant on MORE than one variable.

Example: $w = xyz$ where $\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$

3 The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example: Suppose F=f(x,y,z,t) where x=x(u,v,w), y=y(u,v,w), z=z(u,v,w), and t=t(u,v,w),

Find: (show tree diagram)

Using the chain rule to generate a formula as an alternate to implicit differentiation.

Recall example from 14.3 that we did using implicit differentiation: **Given** $x^2 + y^2 + z^2 = 1$, **find** $\frac{\partial z}{\partial y}\left(\frac{2}{3}, \frac{1}{3}\right)$

Assuming z can be expressed as a function of f(x,y) then we should be able to find $\frac{\partial z}{\partial y}$, but rater than solve for z (explicit) or take the partial with respect to y of both sides (implicit) we will introduce a new function,

$F(x,y,z) = x^2 + y^2 + z^2$ and represent the given surface $x^2 + y^2 + z^2 = 1$ as a particular level surface of F, $F(x,y,z) = 1$ (this is a common technique as we go on)

Now F is a function of x, y, and z……… where z is a function of x and y
(That is F(x,y,z) where z=f(x,y)) which means that in the words used earlier, F is ultimately a function of x and y.

Then by the chain rule:
$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$ and $\frac{\partial F}{\partial y} = \frac{\partial F}{\partial x}\frac{dx}{dy} + \frac{\partial F}{\partial y}\frac{dy}{dy} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}$

Applying this to a surface, represented by the equation $F(x,y,z) = k$ we get

So
$\frac{\partial z}{\partial x} =$ and $\frac{\partial z}{\partial y} =$

Thus for our example:

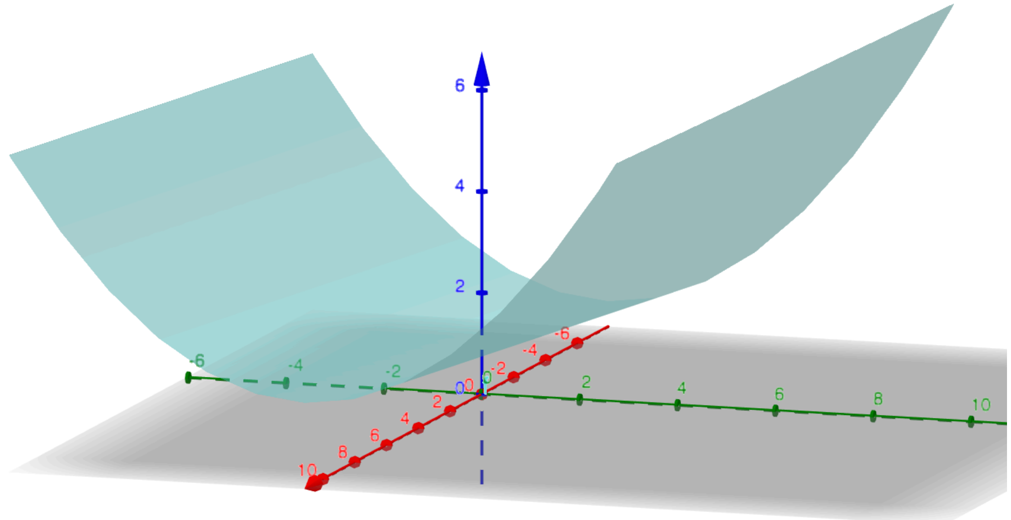
See also example 8 page 982 for R2 version

14.6 The Derivative

From our earlier introduction to derivative, we defined

$$D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}$$

but we could not yet compute it. The chain rule will enable us to compute it.



$z = f(x,y)$ and we found $\begin{cases} x = x_0 + ah \\ y = y_0 + bh \end{cases}$

So z is ultimately a function of h only. Then

$$\frac{dz}{dh} =$$

Introducing gradient notation, define $\vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$ then

$$D_{\vec{u}} f(x,y) =$$

<u>Example</u>: Find the (directional) derivative of $f(x,y) = xy^2 + \ln x$, at point (1,2) in the direction of $\vec{v} = \langle 3,4 \rangle$

Significance of the Gradient

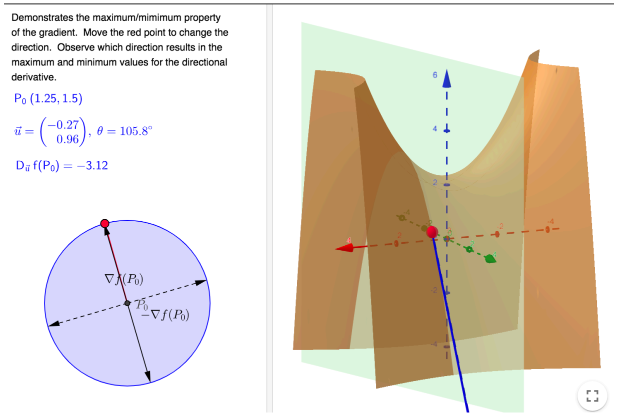
$$D_{\vec{u}} f(x_0,y_0) = \vec{\nabla} f(x_0,y_0) \bullet \vec{u} = \left\|\vec{\nabla} f(x_0,y_0)\right\| \left\|\vec{u}\right\| \cos\theta$$

Maximum value of the directional derivative at (x_0,y_0) occurs in the direction of________
and the value of the derivative in that direction is ________________________

Minimum value of the directional derivative at (x_0,y_0) occurs in the direction of________
and the value of the derivative in that direction is ________________________

Traveling in the direction which is orthogonal to the gradient ________________________

Illustration on 5C page: https://www.geogebra.org/m/tZgrSxQ4#material/vBNTj7Y2



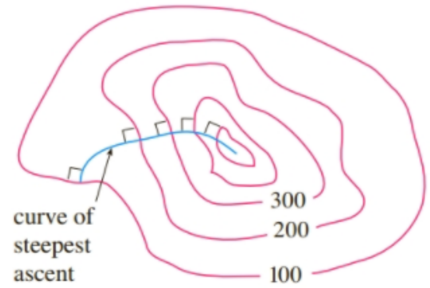


FIGURE 12

Extends to R3

For $f(x,y,z)$, $$\vec{\nabla} f(x,y,z) = \left\langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \right\rangle$$

$$D_{\vec{u}} f(x_0,y_0,z_0) = \vec{\nabla} f(x_0,y_0,z_0) \bullet \vec{u} = \left\| \vec{\nabla} f(x_0,y_0,z_0) \right\| \left\| \vec{u} \right\| \cos\theta$$

As before, the maximum of the directional derivative at (x_0,y_0,z_0) occurs in the direction of the gradient and the minimum occurs in the direction opposite the gradient. Here, the gradient is orthogonal to the level *surface* of $f(x,y,z)$.

EXAMPLE 7 Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

SOLUTION The gradient of T is

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}$$

$$= -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{i} - \frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{j} - \frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{k}$$

$$= \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2}(-x\,\mathbf{i} - 2y\,\mathbf{j} - 3z\,\mathbf{k})$$

At the point $(1, 1, -2)$ the gradient vector is

$$\nabla T(1, 1, -2) = \tfrac{160}{256}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = \tfrac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$$

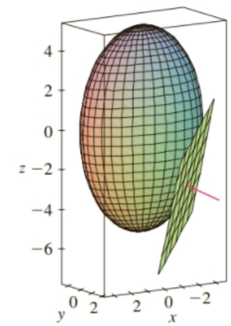
By Theorem 15 the temperature increases fastest in the direction of the gradient vector $\nabla T(1, 1, -2) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ or, equivalently, in the direction of $-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ or the unit vector $(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})/\sqrt{41}$. The maximum rate of increase is the length of the gradient vector:

$$|\nabla T(1, 1, -2)| = \tfrac{5}{8}|-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \tfrac{5}{8}\sqrt{41}$$

Therefore the maximum rate of increase of temperature is $\frac{5}{8}\sqrt{41} \approx 4°\text{C/m}$. ■

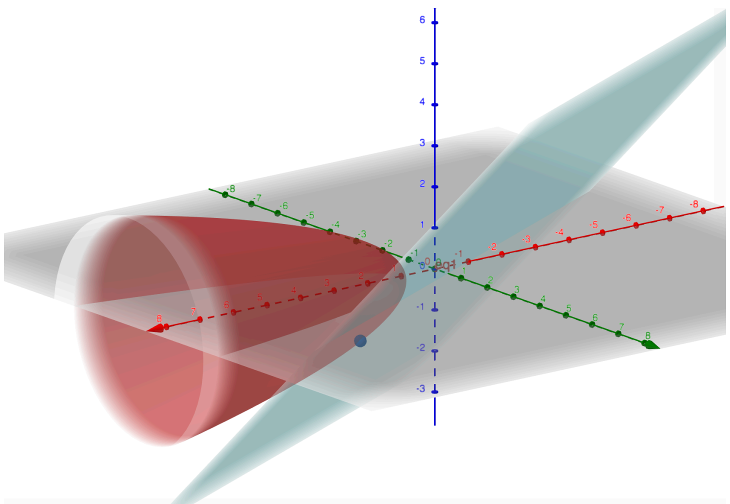
14.6 cont'd : Tangent Planes

We are often interested in finding the plane tangent to a surface at a given point.



As we learned earlier, any surface can be expressed as a level surface of a function of three variables. F(x,y,z)=k. Given the previous discussion, $\vec{\nabla}F(x_0,y_0,z_0)$ is orthogonal to the level surface of F. That will be our normal vector to the plane.

Example: Find the equations of the tangent plane and the normal line to the surface $x=y^2+z^2+1$ at the point (3,1,-1).



14.4 : Tangent Planes and Differentials

Tangent Planes

In 14.6 we learned how to find the equation for a plane tangent to a surface. If we express the surface as a level surface of a function of 3 variables, F(x,y,z)=k, then the normal vector for the tangent plane at the point (x_0,y_0,z_0) is $\vec{n}=\vec{\nabla}F(x_0,y_0,z_0)$.

EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

In section 14.4, your book derives another formula that can be used in the special case that the surface can be expressed as a function, z=f(x,y).

Note the similarity between the equation of a tangent plane and the equation of a tangent line:

$$y - y_0 = f'(x_0)(x - x_0)$$

2 Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

SOLUTION Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$
$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then (2) gives the equation of the tangent plane at (1, 1, 3) as

$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

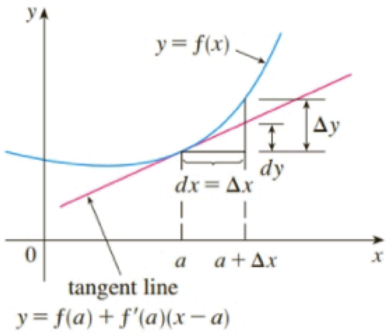
$$z = 4x + 2y - 3$$

■

It is not necessary to remember this formula separately since our method from 14.6 is more general and works in more situations. However, we will use this formula in a derivation which follows.

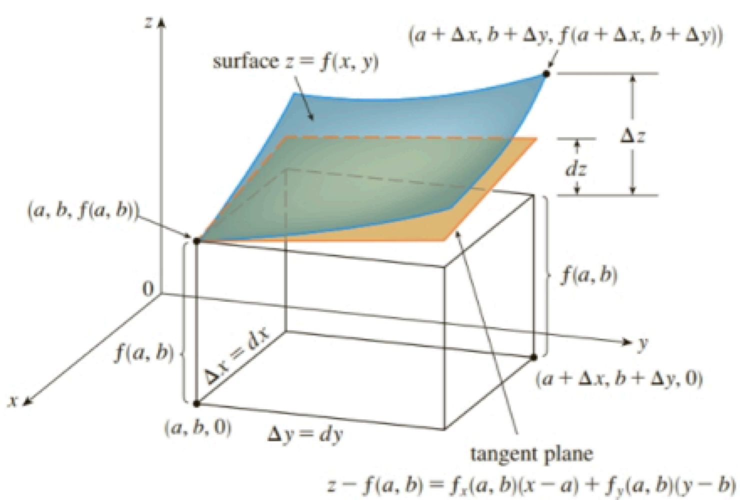
Differentials

Recall from 5A: If y=f(x), the the differential, dy=______________________ What is this giving us.



In section2.9, we used this in two ways. (1) Use dy to approximate Δy, and (2) Approximate functional values $f(x+\Delta x)$

Similarly, for z=f(x,y) we would want dz to represent________________________________



Deriving the formula for the differential dz:

We use differentials in two ways:

(1) Approximate Δz

EXAMPLE 4
(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.
(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

SOLUTION
(a) Definition 10 gives

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = (2x + 3y)\,dx + (3x - 2y)\,dy$$

(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2, 3)\\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(2)(3) - 3^2]\\ &= 0.6449\end{aligned}$$

Notice that $\Delta z \approx dz$ but dz is easier to compute.

The need to approximate Δz comes up in physical applications like that of computing error, see example 5

(2) Approximating functional values $f(a+\Delta x, b+\Delta y)$

Since $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$,

$f(a+\Delta x, b+\Delta y) =$ ________________________________

Example: Use differentials to approximate $\sqrt{9(1.95)^2+(8.1)^2}$

14.7 Extrema of f(x,y)

5A Review problem: Given $f(x) = 3x^4 - 16x^3 + 18x^2$, find:

Critical Numbers: (3.1)

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

---

1) Local Extrema (3.3)

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

---

2) Absolute Extrema

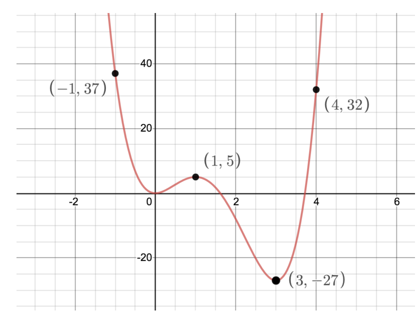
---

3) Absolute Extrema on [-1,4]

(3.1)

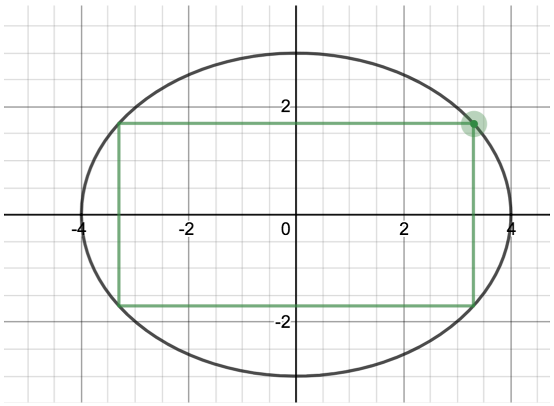
The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.



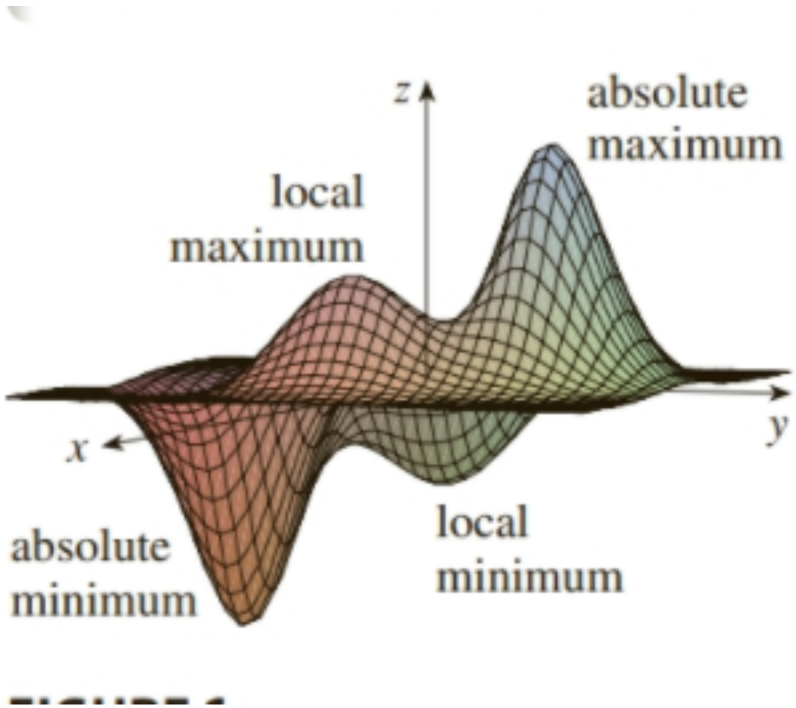
5A review contd: applied problem

Maximize the area of a rectangle inscribed in $\frac{x^2}{16}+\frac{y^2}{9}=1$



Desmos animation (link on 5C page) https://www.desmos.com/calculator/uphhr6aikh

Now Consider Extrema for z=f(x,y)



Observations:

Critical Points:

Example: Find the critical points of $f(x,y)=\frac{1}{100}\left(x^3+y^3-12xy\right)$

14.7i Local Extrema

After computing critical points, what next?

"First Derivative Test"?

"Second Derivative Test"?

How do we even compute a second derivative?

Example: Compute $D^2_{\vec{u}} f(4,4)$ for $f(x,y)=\frac{1}{100}\left(x^3+y^3-12xy\right)$ in the direction of $\vec{u}=\left\langle\frac{3}{5},\frac{4}{5}\right\rangle$

$\vec{\nabla} f(x,y)=\left\langle\frac{1}{100}(3x^2-12y),\frac{1}{100}(3y^2-12x)\right\rangle$, so

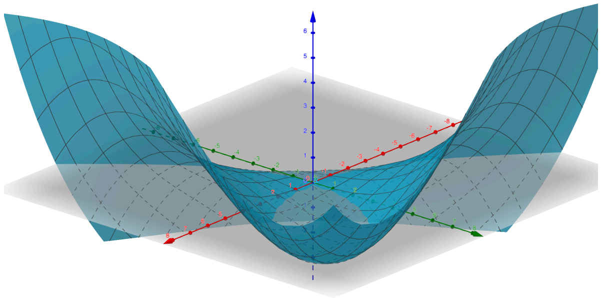
$$D_{\vec{u}} f(x,y)=\vec{\nabla} f(x,y)\bullet\vec{u}=\frac{3}{500}(3x^2-12y)+\frac{4}{500}(3y^2-12x)=\frac{9}{500}x^2-\frac{36}{500}y+\frac{12}{500}y^2-\frac{48}{500}x$$

Now we take the derivative of this function in the direction of $\vec{u}=\left\langle\frac{3}{5},\frac{4}{5}\right\rangle$

$$D_{\vec{u}}\left(D_{\vec{u}} f(x,y)\right)=D_{\vec{u}}\left(\frac{3}{500}(3x^2-12y)+\frac{4}{500}(3y^2-12x)=\right)$$

$$D^2_{\vec{u}} f(x,y)=\frac{1}{2500}(54x-288+96y)$$

$$D^2_{\vec{u}} f(4,4)=$$



How do we show $D^2_{\vec{u}} f(4,4) > 0$ for every direction? (See proof pg 1007)....

Test For Local Extrema

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

So for our example, $f(x,y) = \frac{1}{100}\left(x^3 + y^3 - 12xy\right)$ with critical points (4,4) and (0,0)

$f_x = \frac{1}{100}(3x^2 - 12y)$ $\qquad$ $f_y = \frac{1}{100}(3y^2 - 12x)$

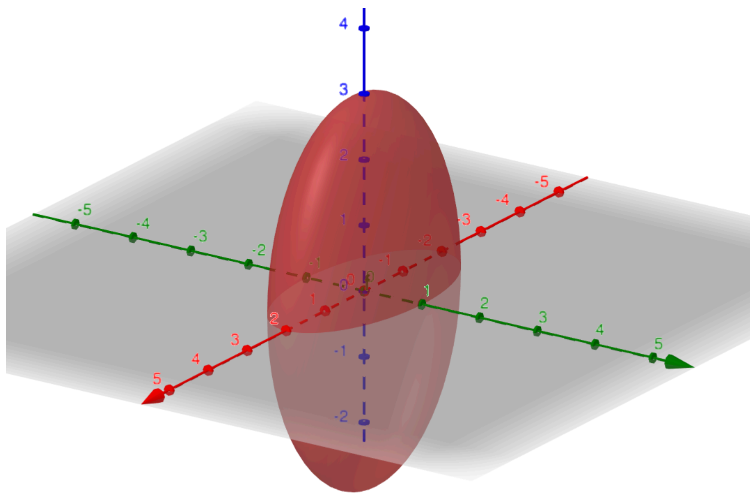
$$D = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} =$$

At (0,0) D=

At (4,4) D=

14.7ii Absolute Extrema

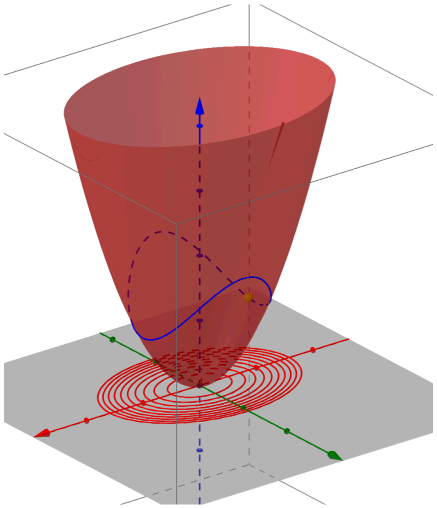
Find the maximum volume of a rectangular box that can be inscribed in the ellipsoid $9x^2 + 36y^2 + 4z^2 = 36$



How do we know this critical point actually yields an ABSOLUTE MAX? MUST VALIDATE THIS IN SOME WAY.

14.7iii Absolute Extrema for f(x,y) Continuous on Closed Domain

Example. Find the absolute extrema of $f(x,y)=x^2+2y^2$ on the closed domain (or "subject to the constraint") $x^2+y^2\le 1$



https://www.geogebra.org/m/RtISr7GW#material/i7ZQsiGf

For another example, see 5C page http://pccmathuyekawa.com/classes-taught/math_5c/file_cabinet/handouts/14.7_HW.jpg
Or video. https://youtu.be/LnX-UZ30ULA

14.7 Summary

14.7i: Local Extrema

- Find the critical points by solving the system $\begin{cases} f_x(x,y)=0 \\ f_y(x,y)=0 \end{cases}$
- For each critical point apply the second derivative test. Compute D= $\begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix}$
  - If D>0, there is a local extremum, to determine if it is a max or min find f_{xx} (or f_{yy}) at the critical point
    - If f_{xx}>0, think concave up, so there is a local min.
    - If f_{yy}<0, think concave down, so there is a local max
  - If D<0, there is not a local extremum at that point. This yields a saddle point.

14.7ii: Absolute Extrema subject to a constraint equation

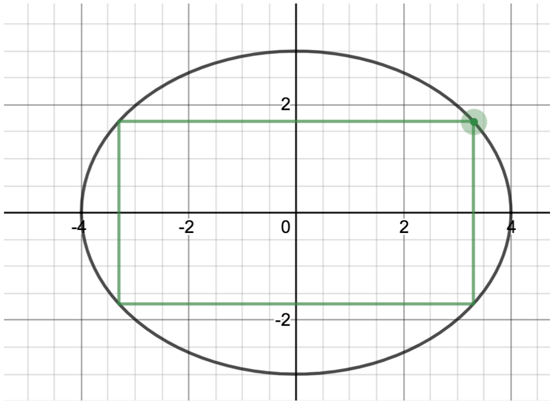
- Incorporate the constraint into the function you wish to optimize to create a function of two variables f(x,y).
- Find the critical points by solving the system $\begin{cases} f_x(x,y)=0 \\ f_y(x,y)=0 \end{cases}$
- Validate whether this critical point actually yields an absolute extremum. Often we do this using physical vadiation.
- Make sure to answer the question asked. Is the max value asked? The input? Both?

14.7iii: Absolute Extrema: Special case f(x,y) continuous on closed domain.

- Compare values of f(x,y) both at critical points and on the boundary of the domain.
- Find the critical points by solving the system $\begin{cases} f_x(x,y)=0 \\ f_y(x,y)=0 \end{cases}$, the find f at those critical points which are in the domain.
- Consider the boundary D (if D is piecewise smooth, repeat this step for each piece of the boundary).
  - Incorporated the boundary curve(s) into f(x,y) to create a function of one variable, say g(x). (or it could be a function of y)
  - Find the domain interval for the input interval. $a \le x \le b$ (or $a \le y \le b$)
  - Treat as a 5A closed interval method problem (3.1) and find the abs . max for that f(x) on [a,b]. Compare the values you get here to the value of f at critical numbers

14.8 Lagrange Multipliers- A method for Optimizing a Function subject to a constraint equation (Omit two constraint problem)

Example motivating the method of Lagrange Multipliers: Maximize the area of a rectangle inscribed in $\frac{x^2}{16}+\frac{y^2}{9}=1$

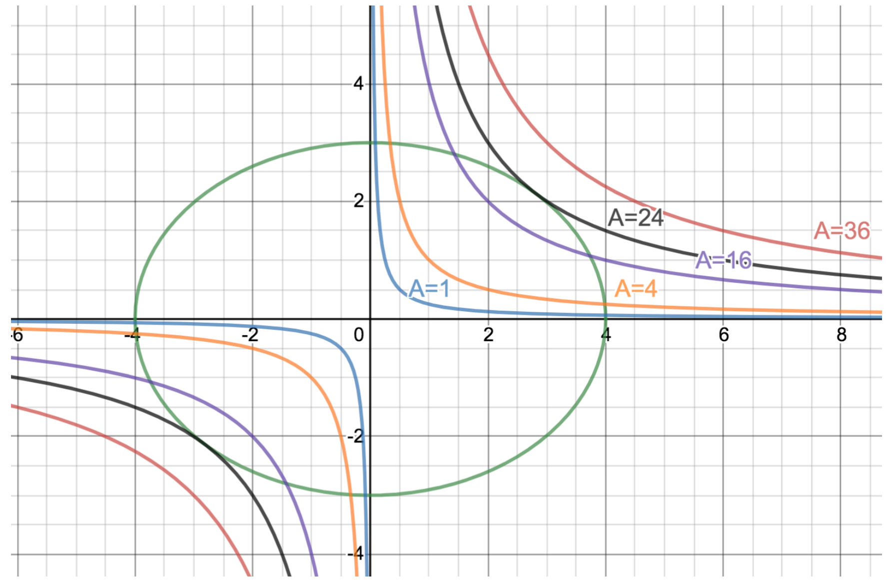


Desmos animation (link on 5C page) https://www.desmos.com/calculator/uphhr6aikh

Method of Lagrange Multipliers:

To optimize f subject to a constraint equation g=k,

$$\begin{cases} \vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y) \\ g(x,y) = k \end{cases} \qquad \begin{cases} \vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z) \\ g(x,y,z) = k \end{cases}$$



Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and $$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Redo First Example: Maximize $A(x,y) = 4xy$ subject to constraint equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Redo Example from last section:

Find the maximum volume of a rectangular box that can be inscribed in the ellipsoid $9x^2 + 36y^2 + 4z^2 = 36$

As discussed previously, we wish to maximize $V(x,y,z) = 8xyz$ subject to the constraint $9x^2 + 36y^2 + 4z^2 = 36; \quad x,y,z > 0$

So our "$f(x,y,z)$"is $V(x,y,z) = 8xyz$ and our

Lagrange Multiplier Illustration https://www.geogebra.org/m/RtISr7GW#material/i7ZQsiGf